Engineering Notes

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Sensitivity of Closed-Loop Eigenvalues and Robustness

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I. Introduction

THIS Note examines how the relative locations of controller (full state feedback) and estimator eigenvalues in a closed-loop linear control system affect the sensitivity of these eigenvalues with respect to uncertain plant parameters. Analytical results indicate that a controller eigenvalue lying near an estimator eigenvalue produces large eigenvalue sensitivity. Both the analytical and numerical results for control of a flexible structure indicate that, for the design of robust control systems, eigenvalue sensitivity should be reduced by separating controller eigenvalues from estimator eigenvalues.

Many matrices have eigenvalues that become highly sensitive to parameter variations in the matrices when two eigenvalues approach one another. The most common example in linear systems is the 2×2 matrix that represents a critically damped mode, where the first-order sensitivity of the eigenvalues with respect to a parameter is infinite. On the other hand, there are matrices whose eigenvalue sensitivities do not become large when eigenvalues approach one another, even though the corresponding eigenvectors approach linear dependence. For a given class of matrices, therefore, there are two questions. Does the first-order sensitivity of eigenvalues with respect to parameters become large as the distance between eigenvalues becomes small? If so, is this only a local characteristic of the dependence of the eigenvalues upon the parameters or does it indicate a large sensitivity over a significant range of parameter values?

This Note addresses these questions for the closed-loop system that results from applying a state-estimator-based compensator to a linear control system. The parameters are uncertain plant parameters. Sections II and III derive formulas for the first-order sensitivities of the closed-loop eigenvalues with respect to the plant parameters. These formulas indicate that, except in rare cases, the sensitivities grow without bound as the distance between a controller eigenvalue and an estimator eigenvalue becomes small.

The example in Sec. IV illustrates the effect of the sensitivity with respect to uncertain plant frequencies on robustness. The numerical results show that the high eigenvalue sensitivity diminishes robustness significantly and that separating the controller and estimator eigenvalues improves robustness. This suggests that first-order sensitivity affects robustness over a significant range of eigenvalue locations.

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II. Control System, Compensator and Closed-Loop Spectrum

The closed-loop system in Fig. 1 satisfies the differential equation

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = A_{c\ell}(\beta) \begin{bmatrix} x \\ \hat{x} \end{bmatrix} \tag{1}$$

where

$$A_{c\ell}(\beta) = \begin{bmatrix} A(\beta) & -B(\beta)F \\ GC(\beta) & [A(\beta_0)-B(\beta_0)F-GC(\beta_0)] \end{bmatrix} \tag{2}$$

and β is a plant parameter. The state $x(t)$ is an n vector, the control $u(t)$ is an r vector, and the measurement $y(t)$ is an m vector. The $n\times n$ matrix A, $n\times r$ matrix B, and $m\times n$ matrix C are all real. The n vector $\hat{x}(t)$ is an estimate of $x(t)$. The gain matrices F and G are determined by some compensator design procedure, for which the value of β is assumed to be β_0.

The following standard similarity transformation is useful here:

$$TA_{c\ell}T^{-1} \begin{bmatrix} [A-BF] & BF \\ 0 & [A-GC] \end{bmatrix} \tag{3}$$

where

$$T=T^{-1}=\begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \tag{4}$$

This transformation shows that, as is well known, the spectrum of $A_{c\ell}$ is the union of the spectra of $[A-BF]$ and $[A-GC]$. We refer to the eigenvalues of $[A-BF]$ as the controller eigenvalues and to the eigenvalues of $[A-GC]$ as the estimator eigenvalues. Also, from here on, *we assume that the eigenvalues of $A_{c\ell}$ are distinct.*

Now we derive some formulas involving closed-loop eigenvectors that will be useful in the next section. We denote by X_e and X_c the $n\times n$ matrices whose columns are the eigenvectors of $[A-GC]$ and $[A-BF]$, respectively, and by Z the $2n\times2n$ matrix whose columns are the eigenvectors of $A_{c\ell}$. Also, Λ_e and Λ_c are the $n\times n$ diagonal matrices containing the eigenvalues of $[A-GC]$ and $[A-BF]$, respectively, and $\Lambda_{c\ell}$ is the $2n\times2n$ matrix

$$\Lambda_{c\ell}=\begin{bmatrix} \Lambda_c & 0 \\ 0 & \Lambda_e \end{bmatrix} \tag{5}$$

It follows from Eqs. (3) and (4) that

$$Z=\begin{bmatrix} X_c & X_c\tilde{X} \\ X_c & [X_c\tilde{X}-X_e] \end{bmatrix} \tag{6}$$

and

$$Z^{-1}=\begin{bmatrix} [X_c^{-1}-\tilde{X}X_e^{-1}] & \tilde{X}X_e^{-1} \\ X_e^{-1} & -X_e^{-1} \end{bmatrix} \tag{7}$$

where the $n \times n$ matrix $\tilde{X}$ satisfies

$$\Lambda_c \tilde{X} - \tilde{X}\Lambda_e = -X_c^{-1} BFX_e \tag{8}$$

There exists a unique solution to Eq. (8) because, by hypothesis, Λ_c and Λ_e have no eigenvalues in common.

III. Sensitivity of the Closed-Loop Eigenvalues with Respect to Plant Parameters

Here, we study the first-order sensitivity of the eigenvalues of $A_{c\ell}(\beta)$ with respect to an error between the true plant parameter β and the nominal value β_0 assumed for compensator design. By standard results,[1,2] we have

$$\Lambda_{c\ell_\beta} = \frac{\partial}{\partial\beta}\Lambda_{c\ell} = \mathrm{diag}[Z^{-1}A_{c\ell_\beta}Z] \tag{9}$$

where diag $[\cdot]$ means the diagonal matrix with the same diagonal elements and

$$A_{c\ell_\beta} = \frac{\partial}{\partial\beta}A_{c\ell} = \begin{bmatrix} A_\beta & -B_\beta F \\ GC_\beta & 0 \end{bmatrix} \tag{10}$$

The subscript β always indicates the partial derivative with respect to β. Using Eqs. (6) and (7) and carrying out the multiplication in Eq. (9) yields

$$\Lambda_{c\ell_\beta}(\beta_0) = \frac{\partial}{\partial\beta}\Lambda_{c\ell}(\beta_0) = \mathrm{diag}\begin{bmatrix} \Gamma_1 & 0 \\ 0 & \Gamma_2 \end{bmatrix} \tag{11}$$

where

$$\Gamma_1 = X_c^{-1}[A_\beta(\beta_0) - B_\beta(\beta_0)F]X_c - \tilde{X}X_e^{-1}[A_\beta(\beta_0) - B_\beta(\beta_0)F - GC_\beta(\beta_0)]X_c \tag{12}$$

$$\Gamma_2 = X_e^{-1}B_\beta(\beta_0)FX_e + X_e^{-1}[A_\beta(\beta_0) - B_\beta(\beta_0)F - GC_\beta(\beta_0)]X_c\tilde{X} \tag{13}$$

According to Eq. (8), the $i-j$ element of the matrix $\tilde{X}$ approaches infinity as the reciprocal of the difference between the ith controller eigenvalue and the jth estimator eigenvalue, except in rare special circumstances. In general, this element of $\tilde{X}$ enters the derivative of each closed-loop eigenvalue according to Eq. (11) and produces the large sensitivity when the estimator and controller eigenvalues are close. Also, when estimator eigenvectors and/or controller eigenvectors are nearly linearly dependent, elements of X_e^{-1} and/or X_c^{-1} approach infinity and produce large sensitivity.

It is possible for a controller eigenvalue and an estimator eigenvalue to approach one another and for the terms in Γ_1 and Γ_2 involving the reciprocal of the difference of these eigenvalues to be zero, so that the large eigenvalue sensitivity does not arise. However, this can happen only in rare instances. The analysis to prove this claim is beyond the scope of this Note, but results such as the following can be proved with Eqs. (8), (12) and (13): *if* $B_\beta = 0$, $C_\beta = 0$, *S is the space of triples* (B,F,A_β) *of appropriate dimension and* S_0 *is the set of triples* (B,F,A_β) *for which all diagonal elements of* Γ_1 *remain bounded as the gain matrix G is varied to make an estimator eigenvalue approach a controller eigenvalue, then* S_0 *has measure zero as a subset of S.* The idea is that, for given elements of the product involving $\tilde{X}$ in Eq. (12) to be zero, (B,F,A_β) must lie in some manifold in S. That the dimension of this manifold is less than the dimension of S follows from the fact that X_c and X_e are nonsingular. The proof requires some tedious arguments involving the inverse function theorem because Λ_c and X_c vary with B and F.

That the large eigenvalue sensitivity indicated by Eqs. (12) and (13) arises from the particular form of the closed-loop matrix in Eq. (2), and not from a universal property of matrices that implies large eigenvalue sensitivity whenever two eigenvalues approach one another can be seen by replacing β_0 with β in Eq. (2), and replacing the zero in Eq. (10) with the appropriate derivatives. The additional terms that then appear in Eqs. (12) and (13) cause the terms involving $\tilde{X}$ to vanish. Thus, if the parameter β varies in the compensator in the same way that it varies in the plant, no large sensitivity results when controller and estimator eigenvalues are close or even equal.

Equations (12) and (13), along with Eq. (8), provide the most efficient algorithm we know for computing the sensitivities of all the eigenvalues of the $2n \times 2n$ matrix $A_{c\ell}$. In general, the number of arithmetic operations required for computation of all eigenvalue sensitivities for a $2n \times 2n$ matrix is proportional to $(2n)^3$, but with the results here the required number of operations is proportional to $2n^3$.

IV. Example

This section illustrates the effect of eigenvalue sensitivity on robustness. The example is taken from Ref. 3 where a four-mode model of flexible structure is used. The structure consists of a rotating hub, flexible beam, and tip mass. There is one input—the control torque—and two outputs—the rigid-body angle and tip-mass deflection.

The modal equations have the form of the plant in Fig. 1, with

$$A(\beta) = \begin{bmatrix} 0 & I \\ -\beta\Omega^2 & -c_0\Omega^2 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ B_0 \end{bmatrix},\ C = \begin{bmatrix} B_0^T & 0 \\ C_2 & 0 \end{bmatrix} \tag{14}$$

$$\Omega = \begin{bmatrix} 0.0 & & & \\ & 0.96571 & & \\ & & 3.23510 & \\ & & & 7.35588 \end{bmatrix} \tag{15}$$

$$B_0^T = [0.00775 \quad -0.03664 \quad -0.05755 \quad -0.05214] \tag{16}$$

$$C_2 = [0.0000 \quad 4.40247 \quad 6.12283 \quad 5.86388] \tag{17}$$

where Ω is a 4×4 diagonal matrix containing the natural frequencies of the model, $c_0 = 10^{-3}$ is the damping coefficient, and β is an uncertain parameter with nominal value $\beta_0 = 1$. The first element of Ω is zero, corresponding to the rigid-body mode. When we refer to the natural frequencies of the structure, we will mean the three nonzero elements of Ω only. We assume that matrices B and C do not depend on β.

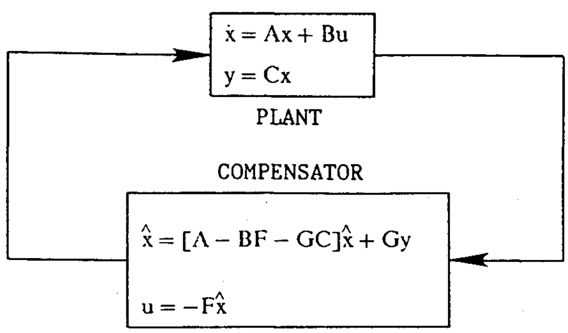

Fig. 1 Closed-loop system.

Of course, this model may not be sufficiently accurate for designing a compensator for the real structure. In Refs. 4–6, we have studied the question of how accurate a finite-element model must be for compensator design and how many modes must be represented in the estimator. While robustness with respect to truncation errors is as important as robustness with respect to parameter errors, we assume here that the four-mode model is the structure to best illustrate the effect on robustness of the eigenvalue sensitivity discussed in the previous section.

For our four-mode model of the structure, we designed a family of linear quadratic Gaussian (LQG) compensators.[7] Each compensator has the control gain

$$F=R_c^{-1}B^TP_c \tag{18}$$

Table 1 Closed-loop eigenvalues with robust compensator

Eigenvalues of $[A(\beta_0)-BF]$	Eigenvalues of $[A(\beta_0)-GC]$
$-0.4221\pm i0.5805$	$-0.5347\pm i0.1362$
$-0.5915\pm i1.0571$	$-1.2888\pm i2.2618$
$-0.6861\pm i3.3011$	$-2.2686\pm i5.7000$
$-0.6773\pm i7.3835$	$-12.914\pm i13.902$

where the matrix P_c satisfies the Riccati equation

$$P_c[A(\beta_0)+I\alpha_c]+[A(\beta_0)+I\alpha_c]^TP_c$$
$$-P_cBR_c^{-1}B^TP_c+Q_c=0 \tag{19}$$

The matrix (scalar in this case) R_c penalizes the control in the standard quadratic performance index and the matrix Q_c penalizes the state. The positive scalar α_c guarantees that the eigenvalues of $[A(\beta_0)-BF]$ (the controller eigenvalues) have real parts to the left of $-\alpha_c$. The control gain for all compensators is computed with

$$\alpha_c=0.2 \tag{20}$$

$$R_c=0.01 \tag{21}$$

and Q_c such that

$$x^TQ_cx=500\,\theta^2+2\,[\text{total energy}] \tag{22}$$

Total energy means kinetic energy plus elastic strain energy in the structure.

The compensators differ in the estimator gains, which are given by

$$G=P_eC^TR_e^{-1} \tag{23}$$

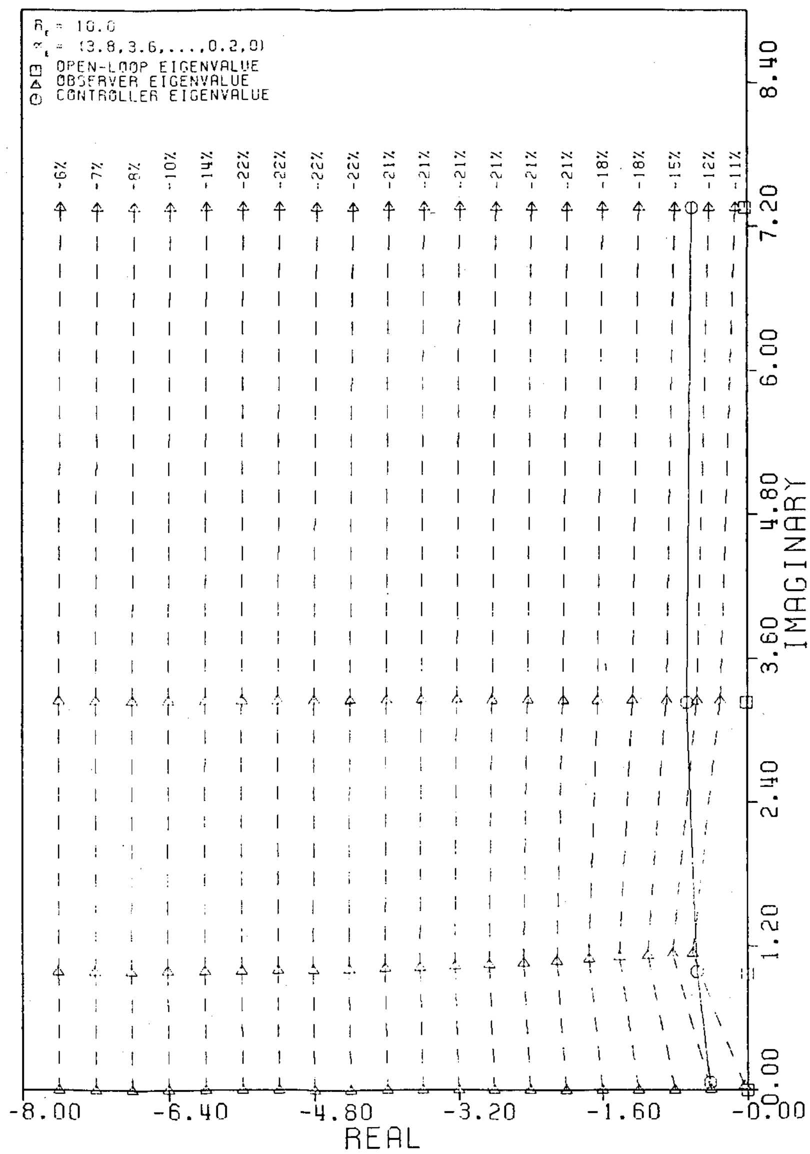

Fig. 2 Robustness test results.

where P_e satisfies the Riccati equation

$$[A(\beta_0)+I\alpha_e]P_e + P_e[A(\beta_0)+I\alpha_e]^T - P_eC^TR_e^{-1}CP_e + Q_e = 0 \qquad (24)$$

Each estimator is a Kalman-Bucy filter for the plant in Fig. 1 with A replaced by $[A(\beta_0)+I\alpha_e]$. A stationary Gaussian process noise with covariance matrix Q_e and a stationary Gaussian measurement noise with covariance matrix R_e are assumed. The positive scalar α_e guarantees that the eigenvalues of $[A(\beta_0)-GC]$ (the estimator eigenvalues) have real parts to the left of $-\alpha_e$. The estimator gains are computed with

$$\alpha_e = \text{variable} = 0.0, 0.2, 0.4, \ldots, 3.8 \qquad (25)$$

$$R_e = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} \qquad (26)$$

$$Q_e = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \qquad (27)$$

(Each block in Q_e is a 4×4 matrix.)

We designed 20 estimators for the values of α_e indicated in Eq. (25) and with each of these estimators, we formed the closed-loop matrix $A_{cl}(\beta)$ in Eq. (2) for a range of β. Our measure of robustness for a compensator is how much β can vary, from the nominal value of 1, before the closed-loop system becomes unstable; i.e., before some eigenvalue of $A_{cl}(\beta)$ has a nonnegative real part. Figure 2 summarizes the results of the robustness test. The solid line connects the eigenvalues of $[A(\beta_0)-BF]$, which are the same for each compensator. (Only eigenvalues with positive imaginary parts are plotted.) For each compensator, a dashed line connects the eigenvalues of $[A(\beta_0)-GC]$ and the number above each of these estimator eigenvalue plots indicates the percentage change in $\sqrt{\beta}$ (from the nominal value of 1) at which the closed-loop system with that compensator becomes unstable. We prefer to look at $\sqrt{\beta}$ because it represents the change in open-loop plant frequencies.

The compensators that place the estimator eigenvalues close to the controller eigenvalues produce a nonrobust closed-loop system, allowing no more than -11% modeling error in the natural frequencies. As the distance between estimator eigenvalues and controller eigenvalues increases, the robustness increases until the compensator will tolerate up to $\pm 22\%$ frequency error and maintain a stable closed-loop system. We have found that the most robust compensator represented in Fig. 2 will also tolerate up to $\pm 22\%$ error in any one of the three plant frequencies when the others remain at their nominal values. It is important to note that the robustness increases as the estimator eigenvalues move away from the controller eigenvalues, even though the performance also increases in the sense that estimator errors decay at faster exponential rates.

Eventually, for $\alpha_e > 2.6$, the robustness starts to decrease again. Close examination of our numerical results indicates that the estimator eigenvectors approach linear dependence for the largest values of α_e, so that large terms enter the right sides of Eqs. (12) and (13) in the matrix X_e^{-1}. This is another demonstration of the relationship between robustness and sensitivity of closed-loop eigenvalues with respect to parameter errors.

In general, as the real part of a conjugate pair of complex eivenvalues becomes large negatively, the corresponding conjugate pair of eigenvectors become nearly linearly dependent. In our example, this happens first for the eigenvalues nearest the real axis, whose frequency is between 0.035 and 10^{-6} rather than zero, as the graph might suggest. It also happens, to a lesser extent, for the pair of eigenvalues with frequency of approximately 1.

Another reason that the robustness cannot be improved more just by moving all of the estimator eigenvalues farther to the left is that the second-order eigenvalue sensitivities with respect to the uncertain parameter involve the reciprocal of the difference of any two estimator eigenvalues and of any two controller eigenvalues. Because this follows from standard formulas[1,2] and is not a result of the special structure of the closed-loop system matrix A_{cl}, we do not discuss it in detail here. Also, we have found the first-order sensitivities to be more important for robustness. However, the pairs of controller and estimator eigenvalues near the real axis cause large second-order sensitivitiy in the closed-loop eigenvalues.

To reduce both the first-order sensitivity produced by almost linearly dependent estimator eigenvectors and the second-order sensitivity produced by closed-loop eigenvalues near the real axis, we designed a new compensator with Q_c, R_c, α_c, Q_e, R_e, and α_c chosen to separate the higher-frequency controller and estimator eigenvalues more than the lower-frequency ones (see Ref. 3 for details). The resulting closed-loop eigenvalues are shown in Table 1. With this compensator, the closed-loop system first becomes unstable at $\sqrt{\beta} = -50\%$, as opposed to -22% for the most robust compensator represented in Fig. 2.

V. Conclusions

The numerical results for the example illustrate the significant effect that the closed-loop eigenvalue sensitivity derived in Sec. III has on robustness with respect to modeling errors. The results in Sec. III suggest and the example confirms that controller and estimator eigenvalues should be separated for a robust design. Almost linearly dependent estimator eigenvectors or controller eigenvectors also diminish robustness.

In the example, we chose to move the estimator eigenvalues to the left of the controller eigenvalues. While such relative placement of controller and estimator eigenvalues is used frequently in compensator design so that the faster decaying estimator error will make the compensator approximate full-state feedback, we have seen no mention in the literature of the relationship demonstrated here between controller/estimator eigenvalue location and robustness. We have found that, to improve robustness by reducing closed-loop eigenvalue sensitivity, the eigenvalue separation may be achieved as well by placing some or all of the controller eigenvalues sufficiently to the left of nearby estimator eigenvalues or, not surprisingly, by separating imaginary parts of eigenvalues. This is important in controlling complex flexible structures, which often have lightly damped modes along with heavily damped modes, making it impractical to place all estimator eigenvalues to the left of all controller eigenvalues.

Although the analysis in Sec. III and the example in Sec. IV deal with a single uncertain parameter, it should be clear that the results apply to any number of parameters. The formulas in Sec. III give the sensitivities of the closed-loop eigenvalues with respect to each parameter. Recently in Ref. 3, we have incorporated the minimization of this sensitivity into the larger problem of integrated control/structure design. The closed-loop eigenvalue sensitivities with respect to all uncertain parameters are included in the overall control/structure objective functional for a numerical optimization problem.

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